Estimating expected losses and liquidity discounts implicit in debt prices

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This paper provides an empirical implementation of a reduced form credit risk model that incorporates both liquidity risk and correlated defaults. Liquidity risk is modeled as a convenience yield and default correlation is modeled via an intensity process that depends on market factors. Various different liquidity risk and intensity process models are investigated. Firstly, the evidence supports a non-zero liquidity premium that is firm specific, reflecting idiosyncratic and not systematic risk. Secondly, the credit risk model with correlated defaults fits the data quite well with an average R^2 of 0.87 and a pricing error of only 1.1%.

1 Introduction

Given the recent exponential growth in the credit derivatives market (see Risk, 2000) and the regulatory induced need to account for credit risk in the determination of equity capital (see Jarrow and Turnbull, 2000b), credit risk modeling has become a topic of current and paramount interest. Although credit risk pricing theory has exploded (see Jarrow, 1998 for a review), the empirical estimation of these models has lagged behind (see Duffie and Singleton, 1997; Madan and Unal, 1998; Duffee, 1999; and Duffie, Pedersen, and Singleton, 2000). To help rectify this imbalance, this paper provides a comprehensive empirical implementation of a reduced-form credit risk model that includes both liquidity risk and correlated defaults. The reduced-form credit risk model implemented is that contained in Jarrow (2001) where a liquidity discount is modeled as a convenience yield and correlated defaults arise due to the fact that a firm's default intensities depend on common macro-factors.

The data used for this investigation is the University of Houston's Fixed Income Database consisting of monthly bid prices taken from Lehman Brothers over May 1991 to March 1997. Twenty different firms' debt issues are investigated where the firms are chosen to stratify various industry groupings.

Five different liquidity premium models were investigated differing in their dependence on various market-wide variables including the spot interest rate, the return on an equity market index, and the equity market index's volatility. These variables were chosen to capture systematic market risks related to interest rates, equities, and the market's volatility. Similarly, the intensity process was allowed to be dependent on the spot rate of interest and the cumulative return on an equity market index.

Overall, the evidence supports the model quite well. First, the best performing liquidity premium model appears to be firm specific and not dependent on market-wide variables. This result is consistent with liquidity risk reflecting only firm specific/idiosyncratic and not systematic risk. Second, the best fitting reduced form credit risk model fits the data quite well with stationary estimated parameters, an average R^2 of 0.87, and an average percentage pricing error of only 0.011.

The previous literature estimating reduced form credit risk models include Duffie and Singleton (1997), Madan and Unal (1998), Duffee (1999), and Duffie, Pedersen, and Singleton (2000). Duffie and Singleton (1997) estimate swap spreads, Madan and Unal (1998) estimate yields on thrift institution certificates of deposit, and Duffie, Pedersen, and Singleton (2000) estimate credit and liquidity spreads for Russian debt. Duffee's (1999) paper is closest to our approach. Using the same bond data, he estimates a reduced form credit risk model where both the default intensity and the default free term structure follow a square root process. The default intensity also depends on the spot rate of interest, so his model captures correlated defaults. Our paper differs from Duffee (1999) in three ways: (i) we use Guassian processes for the default intensity and the default free term structure, (ii) our default intensity has an additional factor – it also depends on the cumulative excess return per unit of risk on an equity market index, and (iii) we explicitly model liquidity risk. Our observation period and firm sample also significantly differ from that in Duffee (1999).

An outline of this paper is as follows. Section 2 introduces both the notation and the reduced form credit risk model. Section 3 provides a description of the data. The parameter estimation is performed in section 4. Section 5 tests the time series stationarity of the parameter estimates, section 6 provides an analysis of the expected loss parameters, while section 7 studies the relative performance of the five different liquidity discount models. Section 8 discusses the absolute performance of the credit risk model studied, while section 9 concludes the paper.

2 The model structure

This section introduces the notation and briefly summarizes the reduced form credit risk model contained in Jarrow (2001). Trading can take place anytime during the interval $[0, \bar{T}]$. Let $\{(\Omega F_{\bar{T}}, P), (F_t : t \in [0, \bar{T}])\}$ be a filtered probability space satisfying the usual conditions.[1] This filtered probability space represents the underlying randomness and information generated in the economy. Traded are default-free zero-coupon bonds and risky (defaultable) zero-coupon bonds of all maturities. Markets are assumed to be frictionless with no arbitrage opportunities, but they can be incomplete with illiquidities present.

Let $p(t,T)$ represent the time t price of a default-free dollar paid at time T where $0 \le t \le T \le \bar{T}$. The instantaneous forward rate at time t for date T is defined by $f(t,T) = -\partial \log p(t,T)/\partial T$. The spot rate of interest is given by $r(t) = f(t,t)$.

Consider a firm issuing risky debt. Let $v(t,T)$ represent the time t price of a promised dollar to be paid by this firm at time T where $0 \le t \le T \le \bar{T}$. The debt is risky because if the firm defaults prior to time T, then the promised dollar may not be paid. Let the random variable τ represent the first time that this firm defaults ($\tau > \bar{T}$ is possible if the firm does not default). Then,

$$N(t) = l_{\{\tau \le t\}} = \begin{cases} 1 & \text{if} \quad \tau \le t \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

denotes the point process indicating whether or not default has occurred prior to time t. We assume that this point process has an intensity $\lambda(t)$ with respect to the given filtration.[2] The time t intensity process, $\lambda(t)\Delta$, gives the approximate probability of default for this firm over the time interval $[t, t + \Delta]$.[3]

If default occurs, we let the zero-coupon bond receive a *fractional recovery* of $\delta(\tau)v(\tau^-,T)$ dollars where $0 \le \delta(\tau)$ and τ^- represents an instant before default.

Under the assumption of no arbitrage, standard arbitrage pricing theory implies that there exists a probability Q equivalent to P such that[4]

$$p(t,T) = E_t\left(e^{-\int_t^T r(u)\mathrm{d}u}\right) \tag{2}$$

and

$$v(t,T) = 1_{(t<\tau)} E_t\left(\delta(\tau)v(\tau-,T)e^{-\int_t^\tau r(u)\mathrm{d}u} 1_{(t<\tau\le T)} + 1e^{-\int_t^T r(u)\mathrm{d}u} 1_{(t<\tau)}\right) \tag{3}$$

where $E_t(\cdot)$ is conditional expectation with respect to Q at time t.

The risky debt value is composed of two parts. The first part is the present value of the promised payment in default. The second part is the present value of the promised payment if default does not occur. Duffie and Singleton (1999) show that expression (3) can be alternatively written as (3a):

$$v(t,T) = 1_{(t<\tau)} E_t\left(e^{-\int_t^\tau [r(u)+(1-\delta(u))\lambda(u)]\,\mathrm{d}u}\right) \tag{3a}$$

This expression shows that the risky zero-coupon's value can alternatively be computed by taking the discounted expectation of the promised dollar, discounting at a rate augmented by the expected loss $(1 - \delta(u))\lambda(u)$ per unit time. As pointed out by Duffie and Singleton (1999), it is important to emphasize that expression (3a) enables only the estimation of the expected loss and not its separate components.

In this empirical investigation, almost all of the US government debt and all the corporate debt studied are coupon bearing. Consequently, we need to price coupon-bearing bonds. First, for the US government debt, a coupon bond is defined to pay coupons of C_j dollars at times t_j for $j = 1,\ldots, n$ where $t_n + T$ is the maturity date. The last coupon at the maturity date is assumed to include the principal repayment. Let the time t price of this default free coupon bond be denoted by $B(t, T)$. Standard no arbitrage arguments give the price of the default free coupon bond as a portfolio of default free zero-coupon bonds, ie,

$$B(t,T) = \sum_{j=1}^{n} C_{t_j} p(t, t_j) \tag{4}$$

Next, consider a risky coupon-bearing bond. Using similar notation, except for the bond's price which will be denoted by a script $\mathcal{B}$, the risky coupon bond is defined to pay coupons of C_j dollars at times t_j for $j = 1,\ldots, n$ where $t_n = T$ is the maturity date. The coupon bond is risky because if the firm defaults prior to the maturity date, the remaining coupons (and principal) may not be paid in full. In default, we assume that the coupon bond is worth the fractional recovery amount of $\delta(\tau)\mathcal{B}(\tau^-, T)$. Other recovery rate assumptions are possible (see Jarrow and Turnbull, 2000a).

Under this recovery rate structure, the value of a risky coupon-bearing bond at time t, denoted by $\mathcal{B}(t, T)$, is equivalent to the cost of the following portfolio of risky zero-coupon bonds:

$$\mathcal{B}(t,T) = \sum_{j=1}^{n} C_{t_j} v(t, t_j) \tag{5}$$

The coupon bond prices in both expressions (4) and (5) are for bonds trading in perfectly liquid markets. Although this is a reasonable approximation for US government debt, it is not so for US corporate debt. Thus, we need to introduce an adjustment for liquidity risk in the pricing of corporate debt.

Let $\mathcal{B}_l(t, T)$ denote the price of an otherwise identical risky coupon bond trading in an illiquid market. The subscript l indicates the bond's price in an illiquid market. In an illiquid and incomplete market, Jarrow (2001) shows that there exists a stochastic process[5] $\gamma(t, T)$ such that

$$\mathcal{B}_l(t,T) = \mathrm{e}^{-\gamma(t,T)}\mathcal{B}(t,T) \tag{6}$$

The argument is that when there are shortages, the risky bond cannot be shorted,[6] and hence $\mathcal{B}_l(t, T) > \mathcal{B}(t, T)$ is possible. The reverse case occurs when there is an oversupply. The process $\gamma(t, T)$ has the interpretation of being a convenience yield for holding the risky debt. When there is a shortage and one cannot readily buy the risky bond, then $-\gamma(t,T) \geq 0$. When there is a glut and one cannot readily sell the risky bond, then $-\gamma(t,T) \leq 0$. In this context, liquidity risk is analogous to a convenience yield from holding an illiquid bond in one's portfolio. The convenience yield is sometimes positive or negative, depending upon market conditions.

To obtain an empirical formulation of the above model, more structure needs to be imposed on the stochastic nature of the economy. Following Jarrow (2001) we consider an economy that is Markov in two state variables: the spot rate of interest and the cumulative excess return per unit of risk on an equity market index. We next introduce the stochastic evolution of these two state variables.

For the spot rate of interest, we use a single factor model with deterministic volatilities, sometimes called the extended Vasicek model, ie,

Spot rate evolution

$$dr(t) = a_r[\bar{r}(t) - r(t)]dt + \sigma_r dW(t) \tag{7}$$

where $a_r \neq 0$, $\sigma_r > 0$ are constants, $\bar{r}(t)$ is a deterministic function of t chosen to match the initial zero-coupon bond price curve,[7] and $W(t)$ is a standard Brownian motion under Q initialized at $W(0) = 0$. The evolution of the spot rate is given under the risk neutral probability Q.

The second state variable is related to an equity market index, denoted by $M(t)$. The evolution for the equity market index is assumed to satisfy a geometric Brownian motion with drift $r(t)$ and volatility σ_m. The correlation coefficient between the return on the market index and changes in the spot rate is denoted by ϕ.

Market index evolution

$$dM(t) = M(t)\big(r(t)dt + \sigma_m dZ(t)\big) \tag{8}$$

where σ_m is constant, and $Z(t)$ is a standard Brownian motion under Q initialized at $Z(0) = 0$ correlated with $W(t)$ as $dZ(t)dW(t) = \phi dt$ with ϕ a constant.

Our second-state variable is $Z(t)$. We see here that $Z(t)$ is a measure of the cumulative excess return per unit of risk (above the spot rate of interest) on the equity market index.

Given the evolutions of the state variables, we next need to specify their relationship to the bankruptcy parameters, the recovery rate and the liquidity discount. This is the task to which we now turn. First, for the default parameters, we assume that:

Expected loss: a function of the spot rate and the market index

$$(1-\delta(t))\lambda(t) \equiv \max\{a_0 + a_1 r(t) + a_2 Z(t),\ 0\}$$
$$\text{where } a_0 \geq 0 \text{ and } a_1, a_2 \text{ are constants} \tag{9}$$

In this formulation, the expected loss per unit time (ie, the (pseudo) probability of default per unit of time multiplied by one minus the recovery rate) is assumed to be a linear function of the state variables $r(t)$ and $Z(t)$ as long as this linear combination is non-negative, zero otherwise. Note that in this formulation of the expected loss process, the recovery rate is allowed to be a stochastic.

For analytic tractability in the empirical implementation, we drop the maximum operator in expression (9). In this case, as the recovery rate is non-negative, this implies that negative default rates ($\lambda(t) < 0$) are possible. If the likelihood of ($\lambda(t) < 0$) is small, this simplification should provide a reasonable approximation to expression (9). Unfortunately, when the intensity process is negative, the default distribution is no longer a proper probability distribution (see Bremaud, 1981). Nonetheless, given the tractability of the subsequent expressions, and the difficulty of the numerical inversion without a closed form solution, we empirically investigate the validity of this linear approximation.

Given these expressions, it is shown Jarrow (2001) that the default free zero-coupon bond and the risky zero-coupon bond's price can be rewritten as:

$$p(t,T) = \mathrm{e}^{-\mu_1(t,T)+\frac{\sigma_1^2(t,T)}{2}} \tag{10}$$

and

$$v(t,T) = 1_{(t<\tau)}\, p(t,T)\mathrm{e}^{-a_0(T-t)a_1\mu_1(t,T)+(2a_1+a_1^2)\frac{\sigma_1^2(t,T)}{2}} \times \mathrm{e}^{-a_2 Z(t)(T-t)+(1+a_1)a_2\phi\eta(t,T)+(T-t)^3\frac{a_2^2}{6}} \tag{11}$$

where

$$\mu_1(t,T) = \int_t^T f(t,u)\mathrm{d}u + \int_t^T \frac{b(u,T)^2\mathrm{d}u}{2}$$

$$\sigma_1^2(t,T) = \int_t^T b(u,T)^2\mathrm{d}u$$

$$b(u,t) = \frac{\sigma_r\left(1-e^{-a_r(t-u)}\right)}{a_r} \tag{12}$$

and

$$\eta(t,T) = -\left(\frac{\sigma_r}{a_r^3}\right)\left[1-\mathrm{e}^{-a_r(T-t)}\right] + \left(\frac{\sigma_r}{a_r^2}\right)\left[\mathrm{e}^{-a_r(T-t)}\right](T-t) + \left(\frac{\sigma_r}{2a_r}\right)[T-t]^2$$

A direct substitution of these zero-coupon bond price formulae into the coupon bond price expressions (4) and (5) gives the analytical expressions used in this empirical investigation, with one exception. To complete the empirical specification of the risky debt model, we need to specify an explicit functional form for the liquidity premium.

To empirically separate the estimates of the liquidity premium $\gamma(t,T)$ from the expected loss $(1-\delta(t))\lambda(t)$, the time to maturity behavior of the liquidity premium and the expected loss needs to be utilized. First note that if the firm is not in default at time t, then as $T \to t$, all the default related terms in the exponent of the risky zero-coupon bond's price in expression (11) approach zero. This follows because the probability of default by the risky firm goes to zero as $T \to t$, so that $v(t,T) \to 1$. Hence, the expected loss component in the risky-zero coupon bond's price is proportional to time to maturity.

In contrast, the liquidity premium's time to maturity behavior is, in general, not proportional to time to maturity. Indeed, liquidity risk is usually thought of as being determined by factors that are independent of the maturity of the bond, including the size of the bond issue, market sentiment concerning its re-trade value, and the size of institutional holdings. If these beliefs are valid, then the liquidity premium contains a fixed component that is not proportional to time to maturity. To the extent that the liquidity premium contains only this fixed component, the subsequent methodology enables us to empirically separate the liquidity premium from the expected loss. To the extent that this is not true, any time to maturity component of the liquidity premium will be confounded into our estimate of the expected loss.

Based on this discussion, as a joint hypothesis to the empirical methodology, we assume that the liquidity premium is independent of the debt's time to maturity:

Liquidity discount

$$\gamma(t,T) = \gamma_0 + \gamma_1 \sum_{j=t-4}^{t} r(j)/5 + \gamma_2 \sigma_m^2(t) + \gamma_3 \sum_{j=t-4}^{t} \left(\frac{M(j) - M(j-1)}{M(j-1)} \right) \Big/ 5 \qquad (13)$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ are constants.[8]

First, the right side of expression (13) is independent of the time to maturity $(T-t)$. Secondly, the liquidity discount is assumed to be an affine function of three market-wide variables: the five-day average spot rate, the volatility of an equity market index, and the five-day average return on the equity market index. These variables were chosen to capture systematic market risks related to interest rates, equities, and the market's volatility. Although other firm specific variables correlated with debt market liquidity could have been included like the bid/ask spread, volume traded, or volume outstanding, unfortunately, none of this information was available in our bond database. Given this omission, however, the reader should be aware that the liquidity premium estimates obtained might incorporate residual model error. This limited formulation, however, does enable us to investigate whether liquidity risk is either firm specific/idiosyncratic or systematic by testing whether ($\gamma_1 = \gamma_2 = \gamma_3 = 0$).

Substitution of expression (13) into the risky coupon-bond price formula (6) completes the empirical specification of the reduced form credit risk model. As seen, analytic formulas are available for both the default free and risky debt issues. These analytic formulae are the basis for the empirical estimation procedure described in the next sections.

3 Description of the data

The data used for this investigation is the University of Houston's Fixed Income Database. This data consists of monthly bid prices for various fixed income securities, including US Treasuries and US corporate debt. The bid prices are taken from Lehman Brothers trading sheets on the last calendar day in each month. For each security included, various identifying information is also provided including embedded options, seniority status, and whether the bid price is transaction based or matrix priced, see Warga (1999) for additional details.

The time period covered in this study is May 1991 to March 1997. The University of Houston Fixed Income Data terminates after March 1997 and no further updates are available. For the US Treasury securities, all outstanding bills, notes and bonds are included in this data and, therefore, included in this study. Being such a large database (containing over 2 million entries), the potential for data errors is quite large. Indeed, a careful examination of the data confirmed this suspicion. Hence, we filtered the data to remove obvious data errors. We excluded Treasury bonds with matrix prices and inconsistent or suspicious issue/dated/maturity dates and coupons. Lastly, using a median yield filter of 2.5%, we also removed US Treasury debt listings whose yields exceeded the median yield by this percent. After filtering, there are approximately 29,100 US Treasury prices left in the sample set.

For the corporate bond price data, we first excluded all debt issues that contained embedded options (call provisions, extendible bonds, convertible bonds, etc) and that were matrix priced. Matrix prices are linear interpolations of bid prices for other traded issues. These prices are not good approximations to traded prices and therefore omitted from the analysis. These two filters left only bid prices on straight coupon bearing bonds.

From these debt issues, we selected twenty different firms chosen to stratify various industry groupings: financial, food and beverages, petroleum, airlines, utilities, department stores, and technology. Within each industry, the firms were chosen to ensure that at least three debt issues were available sometime during the sample period. Only debt classified as senior, senior debentures, and senior notes are included in the subsequent investigation.

The twenty firms included in this study are provided in Table 1. Their industry association, and the starting and ending date for each of the bond price observations are noted. For each firm, on any particular day in the observation period, a bid price may be missing from the data. For this reason, different firms can have different starting dates and different numbers of bond issues at specific dates in the observation period. The number of distinct bonds available on the

TABLE 1 Details of the firms included in the empirical investigation.

	Ticker Symbol	SIC code	First date used in the estimation	Last date used in the estimation	Bonds	Moody's	S&P
Financials							
Security Pacific Corp	spc	6021	12/31/1991	07/31/1994	7	A3	A
Fleet Financial Group	flt	6021	12/31/1991	10/31/1996	3	Baa2	BBB+
Bankers Trust NY	bt	6022	01/31/1994	04/30/1994	3	A1	AA
Merryll Lynch & Co	mer	6211	12/31/1991	03/31/1997	14	A2	A
Food and beverages							
Pepsico Inc	pep	2086	12/31/1991	03/31/1997	8	A1	A
Coca-Cola Enterprises Inc	cce	2086	12/31/1991	06/30/1994	3	A2	AA–
Airlines							
AMR Corporation	amr	4512	02/29/1992	08/31/1994	2	Baa1	BBB+
Southwest Airlines Co	luv	4512	05/31/1992	03/31/1997	3	Baa1	A–
Utilities							
Carolina Power & Light	cpl	4911	08/31/1992	01/31/1993	3	A2	A
Texas Utilities Ele Co	txu	4911	04/30/1994	03/31/1997	4	Baa2	BBB
Petroleum							
Mobil Corp	mob	2911	12/31/1991	02/29/1996	3	Aa2	AA
Union Oil of California	ucl	2911	12/31/1991	03/31/1997	6	Baa1	BBB
Shell Oil Co	suo	2911	03/31/1992	02/28/1995	5	Aaa	AAA
Department stores							
Sears Roebuck & Co	s	5311	12/31/1991	08/31/1996	7	A2	A
Dayton Hudson Corp	dh	5311	04/30/1993	03/31/1997	2	A3	A
Wal-Mart Stores, Inc	wmt	5331	12/31/1991	03/31/1997	3	Aa3	AA
Technology							
Eastman Kodak Company	ek	3861	01/31/1992	09/30/1994	3	A2	A–
Xerox Corp	xrx	3861	12/31/1991	03/31/1997	4	A2	A
Texas Instruments	txn	3674	10/31/1992	03/31/1997	3	A3	A
INTL Business Machines	ibm	3570	01/31/1994	03/31/1997	3	A1	AA–

Ticker Symbol is the firm's ticker symbol. SIC is the Standard Industry Code. "Bonds" is the number of the firm's different senior debt issues outstanding on the first date used in the estimation. "Moody's" refers to Moody's debt rating for the company's senior debt on the first date used in the estimation. "S&P" refers to S&P's debt rating for the company's debt on the first date used in the estimation.