Modelling Default Risk: A New Structural Approach

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Abstract

This paper provides an alternative approach to the structural credit risk models. The first-passage-time approach extends the original Merton [Journal of Finance 29, 449-470] model by accounting for the fact that the default may occur not only at the debt's maturity, but also prior to this date. Default happens when the firm value process crosses an exhaust barrier. In contrast, this paper defines default as the first time the firm value process crosses a barrier, and the area under the barrier is greater than the exogenous level. This technique is used to price risky debt as an example.

Key words: Credit risk; Structural model; Brownian area
JEL Classification: G12; G13; G33

1 Introduction

Since the early 1990s, the credit derivatives market has exploded to well over $5 trillion reported by the British Banker's Association in 2004. A credit derivative can be defined as a risky financial security whose value is derived from an underlying event. The most basic example of a credit derivative is a risky bond that will pay face value in the case of no default, and some value smaller than the face value if default happens.

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The credit risk modeling literature has been essentially developed in two directions. The first is the structural type model. In structural models, the default time is determined by an underlying process describing the firm value. This approach was first introduced by Black and Scholes (1973), and Merton (1974). In this set up, a default event may occur only at the debt's maturity. If the total value of the firm's asset is less than the face value of its debt, firm defaults and debt holders receive the total value of the firm at maturity. Otherwise, firm does not default, and its liability is repaid in full. The firm's equity is viewed as a European call option on the firm's asset value with a strike price equal to the face value of the debt at maturity. On the other hand, the payoff to the liability holders can be viewed as the face value of the loan less a put option with strike equal to the face value of the debt.

The first-passage-time approach extends the original Merton model by allowing the default to occur not only at the debt's maturity, but also prior to this date. In this setup, default occurs if the firm value crosses a (constant or random) barrier. These models were first introduced by Black and Cox (1976). The asset level which triggers default can be imposed exogenously by Black and Cox (1976), Longstaff and Schwartz (1995) or endogenously by having the shareholders optimally liquidate the firm by Leland and Toft (1996) among others. The firm's equity is modelled as European down and out call option instead of a standard call option. In these models, the threshold is an absorbing state, and default leads to bankruptcy right after the firm's asset value crosses the barrier. However, Gilson et al. (1990) show that almost half of the companies in financial distress avoid liquidation through out-of-court debt restructuring.

The second is the reduced form (also called hazard rate) model. In reduced form models, default time is the first jump time of a point process. These models do not try to explain why default happens, rather they model default explicitly by an intensity or compensator process. This makes the default a totally inaccessible stopping time. The reduced form approach was first introduced by Jarrow and Turnbull (1995), and expanded by Duffie and Singleton (1999), and Lando (1994). Hazard rate models are tractable and have better empirical performance because of their reduced form.

In all of the structural models, the underlying assumption is that the firm value process is observable. This makes default time a predictable stopping time with respect to the reference filtration, because of Brownian Motion being a continuous process. This restriction was relaxed by Zhou (2001) by valuing firms with jumps, and Duffie and Lando (2001), which is an information based model. They assume managers observe the true value of the firm, while investors only observe a noisy firm value. Default time is now totally inaccessible, and allows intensity. This transforms the structural model to a reduced form model. This way one reconciles both structural and reduced form models within one framework. Giesecke and Goldberg (2004) and Çetin et al. (2004) are recent examples of information based models. In CJPY (Çetin et al., 2004), default happens the first time a firm's cash flows remain negative for an extended period of time. In the paper, one can also redefine default as the first hitting time to the barrier, then liquidation is the first time a firm's cash flows are negative and remain negative for an extended time period. Here, the market observes whether the cash flows are positive, zero, or negative, whereas the firm's manager observes the cash flows. This special case of information reduction not only transforms the structural model to a reduced form model, but also give us the closed form of intensity.

In structural models, the default happens right after the underlying process hits the barrier. To overcome this obstacle, we could use the excursion theory, and define default as the first time process spends constant unit of time below the barrier. This technique has been used in CJPY (Çetin et al., 2004)[1]. A drawback of this approach is the process can stay some constant units of time below the barrier, but it can fluctuate close to the barrier. An example of this is the firm value passing its exhaust level, yet being very close to this level.

In this paper, I propose a new extension to the structural model. The default is triggered the first time that the cumulative area of the process below a threshold barrier exceeds an exogenous level. In CJPY (Çetin et al., 2004), the distress clock is set to zero each time the process hits the barrier, whereas here, the distress clock is not set to zero. The explanation for this is the default probability of a firm being in financial distress before has to be higher compared to a firm which has never been in financial distress. One byproduct of this paper is the separation of default and liquidation. If you redefine default as the first hitting time to a barrier, then liquidation is defined as the first time the cumulative area of the process is above a specified level. Default is a necessary state for liquidation. To illustrate the economic concepts involved, the closed form solution of a risky zero coupon bond is derived.

An outline for this paper is as follows. Section 2 presents the model. Section 3 values a risky zero-coupon bond, while section 4 concludes the paper.

[1]For another application of excursion theory to option pricing see Chesney et al. (1997)

2 The Model

Let us consider a continuous trading economy with the time interval $[0, T]$. In this economy, default free zero coupon bonds and risky zero coupon bonds of all maturities are traded. A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, Q)$ satisfies the usual conditions. The probability Q is an equivalent martingale probability measure under which the normalized prices of the traded securities follow a martingale. The market for these traded securities is assumed to be arbitrage free, but not necessarily complete.

Let V be the asset value of the firm, and described under risk neutral measure with the following simple stochastic differential equation:

$$dV_t = \sigma dW_t,\ V_0 = x \tag{2.1}$$

with $x > 0,\ \sigma > 0$, and where W is a standard Brownian motion on the given probability space.

2.1 New Default Definition

In structural models default time can be defined in three ways. First, as in the first passage time models, default will happen the first time the firm value crosses the barrier b,

$$\tau = \inf\{t > 0 : V_t = b\}. \tag{2.2}$$

Next is the occupational time approach. Here, default occurs when the firm value process spends an exogenously given amount of time ($\bar{b}$) below the barrier (b) for the first time,

$$\tau = \inf\{t > 0 : \int_0^t 1_{\{V_s < b\}} \geq \bar{b}\}. \tag{2.3}$$

The third way is similar to the excursion models. Default occurs the first time the firm value process spends an exogenously given amount of time after hitting the barrier.

If $g_t^b = \sup\{s \leq t : V_s = b\}$ is the last time that the process crosses barrier before time t, then default occurs the first time process spends $\bar{b}$ units of time under the barrier,

$$\tau = \inf\{t > 0 : (t - g_t^b) \geq \bar{b}\}. \tag{2.4}$$

The first definition will not allow the firm to survive once it crosses the barrier, and default is imminent. Whereas, the second definition will allow the firm to survive, and it will count the time under the barrier. If total allowable time under the barrier widens, default occurs. Similarly, the last definition will allow the firm to stay a certain amount of time under the barrier. In the last two definitions, default depends on time spent under the barrier. As pointed out earlier, the firm value process can fluctuate under the barrier, yet be very close to it. This does not necessarily qualify a firm to be in default. Therefore, I propose a new way of defining default.

A firm will be in financial distress when its asset value reaches an exhaust level. It still survives in financial distress for some time. If financial distress worsens, default will be inevitable. The default barrier represents a safety covenant which is a contractual agreement to let bondholders have the right to force reorganization of the firm. Let A_t be the total loss in firm value with respect to an exogenous level at time t, which is the same as saying the cumulative area of the firm value process below the exhaust level b:

$$A_t = \int_0^t V_s 1_{\{V_s < b\}} ds. \tag{2.5}$$

A firm can survive with this loss for some time before it defaults. Here, financial distress is defined in terms of the actual loss in firm value below the barrier, whereas other structural models define financial distress in terms of time spent under the barrier. Given financial distress level A_t at time t, default is defined as

$$\tau = \inf\{t > 0 : A_t > \bar{b}\}. \tag{2.6}$$

It is the first time the firm value stays under the financial distress level, b, and the cumulative sum of the values exceeds exogenous level $\bar{b}$. This is also the difference between the excursion set up, in which the firm's value hits a barrier and stays under the barrier for an extended period of time. It can stay below the barrier, b, in the excursion models, but it can fluctuate close to the barrier. This new definition of default overcomes the obstacle of default right after passing the threshold, and lets the firm survive even if it passes the distress level. It also allows us not to reset the distress clock to zero. The model allows us to retain the financial distress of the firm.

The basic idea can be seen in Figure (1) for exhaust level b equals to zero. If the sum of values below the barrier gets larger, then default occurs. The probability of default, e.g. $P(\tau < t)$, equals to $P(A_t > \bar{b})$. If one has the distribution of this area, then one can price defaultable securities. In the next section, I will briefly describe how to calculate this distribution.

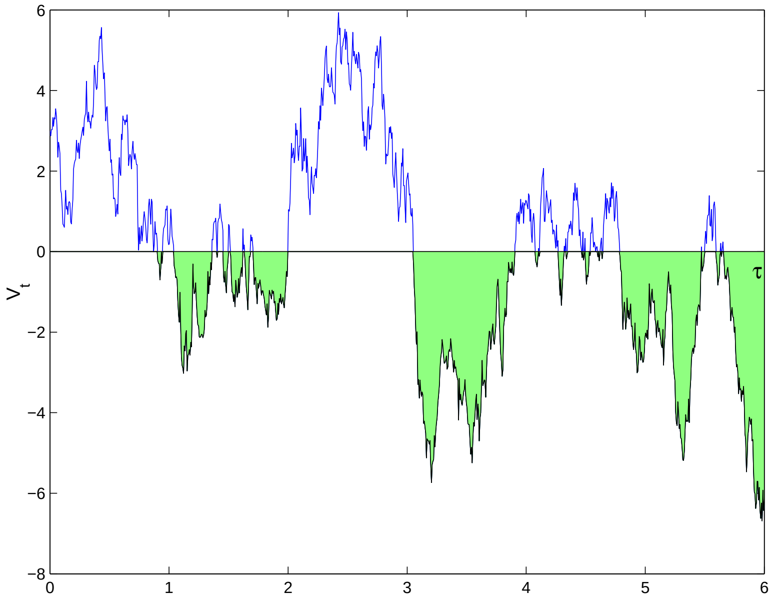

Figure 1: Cumulative area of the process below a zero threshold level

2.2 Distribution of the area of Brownian Motion

The distribution of the area of Brownian motion has been extensively studied for different types of Brownian motions, such as standard Brownian motion, Brownian bridges, excursions and meanders by Shepp (1982), Takác (1993), Perman and Wellner (1995), Durret and Iglehart (1977), Louchard (1984a, 1984b), and Jeanblanc et al. (1997).

In this paper I use a standard Brownian motion case described in Eq. (2.1) for demonstration purposes. One can also apply other types of Brownian motions as in the above literature. In all cases, there are two approaches to find the area distribution. The first is to use the Feynman-Kac formula, which allows us to find the formulas for the double Laplace transform of the distribution of the integral of a functional of Brownian motion. Then these double transforms are used to derive the recurrence formulas for the moment of Brownian integrals, and these in turn yield expansion distributionS of random area in terms of the Laquerre series as in Takác [20]. The second approach is to use excursions to find the double Laplace transforms as in Louchard [15] and [16] using the Airy functions. They describe the probability distribution of the area of Brownian excursion over a unit interval.

Let A_t be the area of standard Brownian motion below level zero,

$$A_t = \int_0^t W_s 1_{\{W_s<0\}} ds. \tag{2.7}$$

The calculation is based on Shepp [19], the first approach mentioned above, and explained in detail in Perman and Wellner [18]. Distribution of the Brownian area, in terms of the Laquerre series, is

$$\begin{aligned} P(A_t \leq \bar{b}) &= P\left(\int_0^t W_s 1_{\{W_s<0\}} ds \leq \bar{b}\right) \\ &= G_\alpha(\beta\bar{b}) + a \sum_{j=0}^{\infty} \frac{c_j}{j} g_{\alpha+1}(\beta\bar{b}) L_{j-1}^{a}(\beta\bar{b}) \end{aligned} \tag{2.8}$$

for $\bar{b} \geq 0$, where $a > 0, \beta > 0$ and

$$c_n \begin{pmatrix} n+a-1 \\ n \end{pmatrix} = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \begin{pmatrix} n+a-1 \\ n-j \end{pmatrix} b^j \mu_j$$

for $n = 0, 1, 2, ...$ As noted in Takács (1993), if we choose

$$a = \frac{\mu_1^2}{\mu_2 - \mu_1^2} \ and \ b = \frac{a}{\mu_1}$$

then the first and second Laquerre coefficients, c_1 and c_2, are both zero, and the next term in the series to enter is the third term. The generalized Laguerre polynomials,

$$L_n^\alpha(\bar{b}) = \sum_{j=0}^{n} (-1)^j \frac{\bar{b}^j}{j!} \begin{pmatrix} n+\alpha \\ n-j \end{pmatrix}, \tag{2.9}$$

defined for $n = 0, 1, 2, ...$ and $\alpha > -1$, are orthogonal on the interval $0 \leq \bar{b} < \infty$ with respect to the density

$$g_{\alpha+1}(\bar{b}) = \frac{e^{-\bar{b}}\bar{b}^{\alpha}}{\Gamma(\alpha+1)} \tag{2.10}$$

and its distribution,

$$G_{\alpha+1}(\bar{b}) = \int_0^{\bar{b}} g_{\alpha+1}(t)dt = \frac{1}{\Gamma(\alpha+1)}\Gamma(\alpha+1, 0, \bar{b}). \tag{2.11}$$

The Laquerre series can now be rewritten in terms of gamma functions,

$$\begin{aligned} L_n^{\alpha}(\bar{b}) &= \sum_{j=0}^{n}(-1)^j \frac{\bar{b}^j}{j!}\begin{pmatrix} n+\alpha \\ n-j \end{pmatrix} \\ &= \sum_{j=0}^{n}(-1)^j \frac{\bar{b}^j}{j!}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}\frac{1}{(n-j)!}. \end{aligned} \tag{2.12}$$

After calculating the Laquerre series and necessary coefficients described above, the area distribution for any values of $\bar{b}$ can be calculated using equation (2.8).

3 Valuation of a Risky Zero-coupon Bond

This section will price a defaultable zero-coupon bond to demonstrate the use of the above default distribution.

Let $(S_t)_{t\in[0,T]}$ denote the price process of a risky zero coupon bond issued by a firm that pays \$1 at time T if no default occurs prior to that date, and zero dollars otherwise. Then, under the no arbitrage assumption, S is given by

$$S_t = E\left[\exp\left(-\int_t^T r_u du\right) 1_{\{\tau>T\}}\right], \tag{3.13}$$

where r_u is the instantaneous interest rate at time u, and E refers to the expectation under the risk neutral probability measure.

To facilitate the computation of expression (3.13), we will assume that interest rates are deterministic, and we have constant recovery δ in the case of default. Let L be the lost rate, then $\delta = 1 - L$. In this case, the price of the risky bond is

$$\begin{aligned} S_t &= E\left[\exp\left(-\int_t^T r_u du\right)\right] E\left[1_{\{\tau > T\}}\right] \\ &= \exp\left(-\int_t^T r_u du\right) \left[1 - LP(\tau < T)\right] \\ &= \exp\left(-\int_t^T r_u du\right) \left[1 - LP(A_t > \bar{b})\right]. \end{aligned} \tag{3.14}$$

We need to calculate $1 - P(A_t \leq \bar{b})$ in the above equation using equation (2.8). As an example, S_t is calculated for $T = 1$ year, $r = 0$, $b = 0$, $L = 1$, and $\bar{b} = 0$ to 0.5 in Fig. 2. This is a special case of zero coupon bond prices, which is identical to one minus the default distribution, e.g. $P(\tau \geq t)$. It is plotted for different levels of the degree of financial distress from 0 to 0.5. If the sum of the values under the barrier increases (the firm's financial distress increases), the risky zero coupon bond price will decrease. Similarly, since the same graph is for survival probabilities, the inverse of this graph can be viewed as default probabilities. Therefore, the default probability will be increasing when the distress level increases, yet in a decreasing fashion. The decrease will be slower for higher financial distress levels. The longer a firm experiences financial distress, the less likely it will be in default.

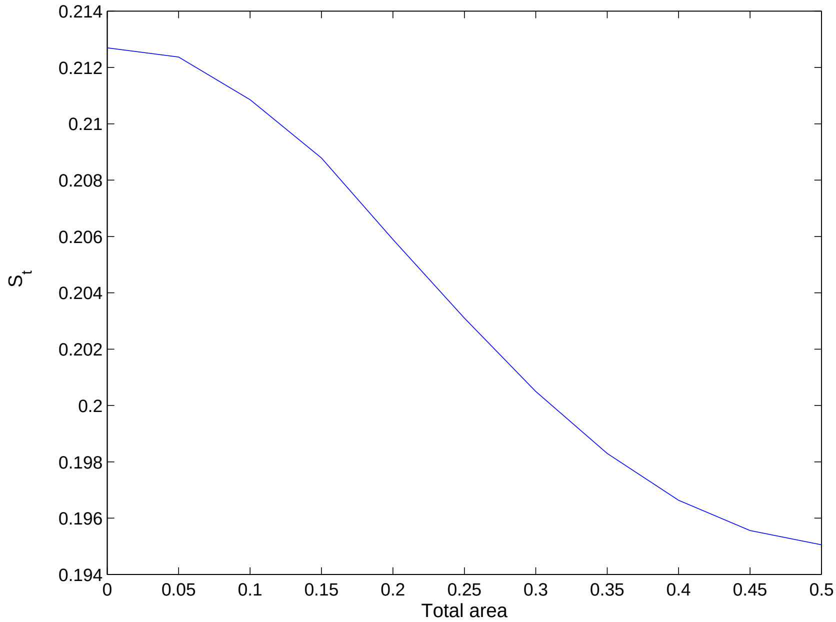

Figure 2: Zero Coupon Bond Prices with different financial stress levels

4 Conclusion

This paper models the default differently from the occupation time and the excursion models. In earlier versions of structural models, we have the problem of triggering the default process whenever the process is below the barrier, and we do not account for the fact that it could be below the barrier but fluctuate very closely to this level. Here, we overcome this obstacle by defining the default in terms of the area under the barrier, and demonstrate this with a simple numerical example. Distribution of default is given using the standard Brownian motion, but it can be extended to other types of Brownian motions such as excursion.

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